Intensity fluctuations in closed and open systems

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We consider the intensity pattern, generated by a monochromatic source, in a disordered cavity coupled to the environment. For weak coupling, and when the source frequency is tuned to a resonance, the intensity distribution $P(I)$ is close to Porter-Thomas distribution. When the coupling increases, $P(I)$ gradually crosses over to the Rayleigh distribution. The joint probability distribution for intensities at two different points is also discussed. [S1063-651X(96)51608-9]

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A wave propagating in a random medium produces a complicated, highly irregular intensity pattern. This pattern is described in statistical terms. One considers an ensemble of different realizations of the random medium and inquires, e.g., about the probability distribution $P(I)$ of the wave intensity I at some point $\mathbf{r}$. In many cases $P(I)$ is accurately described by the Rayleigh distribution

$$P_R(I)=\frac{1}{\langle I\rangle}\exp(-I/\langle I\rangle), \tag{1}$$

where $\langle I\rangle$ is the average intensity. A simple derivation of Eq. (1) is based on an assumption that the field at point $\mathbf{r}$ can be viewed as a sum of many random contributions [1,2]. A more systematic derivation, which paves the way for calculating corrections to the Rayleigh distribution [3,4], is based on the perturbative diagrammatic technique. Here one starts with the wave equation

$$\{\boldsymbol{\nabla}^2+k_0^2[1+\mu(\mathbf{r})]\}\psi_\omega(\mathbf{r})=0, \tag{2}$$

supplemented by appropriate sources. In this equation $\psi_\omega(\mathbf{r})$ describes a field (for instance, pressure field in an acoustic wave or a component of the electric field in an electromagnetic wave), excited by a monochromatic source of frequency ω. The random function $\mu(\mathbf{r})$ describes the fluctuating part of the refraction index and $k_0=\omega/c$, c being the the speed of propagation in the average medium. In the diagrammatic approach one computes moments of the intensity $I\equiv|\psi_\omega(\mathbf{r})|^2$ and reconstructs the distribution $P(I)$.

Let us stress that Eq. (1) applies to the case of a monochromatic wave propagating in an open system. A different type of problem arises if one considers the wave equation (2) in a closed geometry without sources. In this case one inquires about the statistical properties of a single eigenstate $\psi_\alpha(\mathbf{r})$, e.g., about the distribution $P(u)$ of the quantity $u\equiv|\psi_\alpha(\mathbf{r})|^2$. Extensive studies of chaotic [5] and disordered [6] cavities have demonstrated that the main part of the distribution is described by the Porter-Thomas (PT) statistics

$$P_{\rm PT}(u)=\left(\frac{V}{2\pi u}\right)^{1/2}\exp(-uV/2), \tag{3}$$

where V is the volume of the cavity and $\langle u\rangle=1/V$.

The above discussion suggests the following idea. Let us assume that the cavity is weakly coupled to the environment, for instance, via a small opening or due to some small absorption in the bulk. Then, by placing a monochromatic source inside the cavity and tuning its frequency ω to a resonance, one will generate in the cavity an intensity pattern $I(\mathbf{r})$ which closely follows the ''profile'' $|\psi_\alpha(\mathbf{r})|^2$ of the eigenstate α with frequency $\omega_\alpha\simeq\omega$. Thus, the intensity distribution will be given by the Porter-Thomas statistics. On the other hand, for a sufficiently strong coupling (an open system) the distribution should obey the Rayleigh statistics, Eq. (1). The main purpose of this paper is to investigate the crossover between weak and strong coupling and to propose a generalized distribution $P(I)$ which interpolates between the two regimes.

Let us first emphasize the difference between an open and a closed system, using the simple picture of addition of many random waves [1]. In an open system the local field $\psi(\mathbf{r},t)$ can be viewed as a sum of great number of *traveling* waves, arriving at a point $\mathbf{r}$ from various scattering processes:

$$\psi(\mathbf{r},t)=N^{-1/2}\sum_{n=1}^{N}\cos(\theta_n+\mathbf{k}_n\cdot\mathbf{r}-\omega t), \tag{4}$$

where the phases θ_n are completely random and all the amplitudes have been taken to be equal (one could assume random independent amplitudes, without any change in the results). The wave vectors $\mathbf{k}_n$ are uniformly distributed on a d-dimensional sphere ($d=2,3$) of radius k_0. The instantaneous local intensity is defined as $\psi^2(\mathbf{r},t)$. The measured quantity, I, is the intensity averaged over time, i.e., over one period $T=2\pi/\omega$:

$$I=\frac{1}{T}\int_0^T dt\,\psi^2(\mathbf{r},t)$$
$$=\frac{1}{2N}\sum_{n,m}^{N}\cos[\theta_n-\theta_m+(\mathbf{k}_n-\mathbf{k}_m)\cdot\mathbf{r}]. \tag{5}$$

Note that the same expression for I is obtained if one assumes a complex, time independent field

$$\psi(\mathbf{r})=(2N)^{-1/2}\sum_{n=1}^{N}\exp[i(\theta_n+\mathbf{k}_n\cdot\mathbf{r})] \tag{6}$$

and defines the intensity as $I(\mathbf{r})\equiv|\psi(\mathbf{r})|^2$. It follows now from the central limit theorem that both Reψ and Imψ are independent Gaussian variables, with zero mean and equal variances, which leads to Eq. (1) for the intensity distribution.

In a closed system the field is viewed as a sum of many *standing* waves:

$$\psi(\mathbf{r},t)=N^{-1/2}\sum_{n=1}^{N}\cos(\theta_n+\mathbf{k}_n\cdot\mathbf{r})\cos\omega t. \tag{7}$$

As far as the time-averaged intensity is concerned, one can ignore the factor $\cos\omega t$ and define a stationary field

$$\psi(\mathbf{r})=(2N)^{-1/2}\sum_{n=1}^{N}\cos(\theta_n+\mathbf{k}_n\cdot\mathbf{r}). \tag{8}$$

The central limit theorem now tells us that $\psi_\omega(\mathbf{r})$ is a Gaussian variable with zero mean, and the Porter-Thomas statistics

$$P_{\rm PT}(I)=\left(\frac{1}{2\pi\langle I\rangle I}\right)^{1/2}\exp(-I/2\langle I\rangle) \tag{9}$$

for the intensity $I\equiv\psi_\omega^2(\mathbf{r})$ follows immediately.

After clarifying the difference between sums of random traveling waves (open systems) and standing waves (closed systems), we move to the general case of a disordered cavity coupled to the external world. We write the stationary field as

$$\psi(\mathbf{r})=[2N(1+2\epsilon^2)]^{-1/2}\sum_{n=1}^{N}\{\cos(\theta_n+\mathbf{k}_n\cdot\mathbf{r})+\epsilon\exp[i(\theta_n'+\mathbf{k}_n\cdot\mathbf{r})]\} \tag{10}$$

where the parameter ϵ describes the strength of the coupling. For small ϵ, and when the source frequency is tuned to a resonance, the field consists of a large-amplitude standing wave (an eigenstate) with a small traveling wave ''riding'' on top of it. The intensity distribution $P(I)$ is close to the expression in Eq. (9). Large ϵ corresponds to an open system, where the field is mostly a traveling wave, and $P(I)$ is close to the Rayleigh distribution, Eq. (1).

All the phases, θ_n and θ_n' in Eq. (10) are independent and uniformly distributed between 0 and 2π. It is then clear that both Reψ and Imψ are independent Gaussian variables with variances $\langle(\mathrm{Re}\psi)^2\rangle/\langle(\mathrm{Im}\psi)^2\rangle=(1+\epsilon^2)/\epsilon^2$. This leads to the following distribution for the intensity $I=(\mathrm{Re}\psi)^2+(\mathrm{Im}\psi)^2$:

$$P(I)=\frac{1+2\epsilon^2}{2\langle I\rangle\epsilon\sqrt{1+\epsilon^2}}\exp\left[-\frac{I}{4\langle I\rangle\epsilon^2}\frac{(1+2\epsilon^2)^2}{1+\epsilon^2}\right]\times I_0\left[\frac{I}{4\langle I\rangle\epsilon^2}\frac{1+2\epsilon^2}{1+\epsilon^2}\right] \tag{11}$$

where $I_0[x]$ is the modified Bessel function. The nth moment of this distribution is then given by $\langle I^n\rangle=\langle I\rangle^n n!\,{}_2F_1[-n/2,(1-n)/2,1,(1+2\epsilon^2)^{-2}]$, where ${}_2F_1$ is the Gaussian hypergeometric function.

Since the parameter ϵ in Eq. (10) is attached to the propagating part of the field, it can be related to the (averaged over time) current density [7]:

$$\mathbf{J}(\mathbf{r})=\frac{ic}{2k_0}[\psi(\mathbf{r})\nabla\psi^*(\mathbf{r})-\mathrm{c.c}]. \tag{12}$$

Substituting ψ from Eq. (10) and averaging over phases, one finds that $\langle\mathbf{J}(\mathbf{r})\rangle$ vanishes [8] and $\langle J^2(\mathbf{r})\rangle=2c^2\langle I\rangle^2\epsilon^2(1+\epsilon^2)/(1+2\epsilon^2)^2$. Therefore, instead of using the somewhat vague notion of the ''coupling strength ϵ'' for parametrization of the distribution $P(I)$, one can use the dimensionless ratio $\delta\equiv\langle J^2\rangle/c^2\langle I\rangle^2$.

We, thus, propose a one-parameter distribution $P_\delta(I)$ for the intensity (we normalize I to its average value, i.e. choose $\langle I\rangle=1$):

$$P_\delta(I)=\frac{1}{\sqrt{2\delta}}\exp(-I/2\delta)I_0(I\sqrt{1-2\delta}/2\delta). \tag{13}$$

The parameter δ can assume values from 0 (closed system, no current) to 1/2 (open system, maximal current density). When this parameter changes from 0 to 1/2, the intensity distribution changes from Porter-Thomas to Rayleigh.

Let us mention that a somewhat different crossover phenomenon has been considered in Ref. [9]. These authors discussed the statistics of $|\psi_\alpha(\mathbf{r})|^2\equiv u$ for an electron's eigenstate in a quantum dot, in the presence of an arbitrary magnetic field. For zero field the distribution $P(u)$ is given by Eq. (3), whereas for a sufficiently strong field it crosses over to a Rayleigh distribution $P(u)=V\exp(-Vu)$. In this crossover problem, as opposed to the one considered in the present paper, the system always remains closed.

In a similar way one can consider the distribution for the local current density $\mathbf{J}(\mathbf{r})$ or the joint probability distribution $P(I,\mathbf{J})$. We will not discuss here these objects but limit the discussion to the joint probability distribution $P(I_1,I_2)$, where $I_i\equiv I(\mathbf{r}_i)$ $(i=1,2)$. We start with a wave propagating in an open system. The field $\psi(\mathbf{r})$ is then given by Eq. (6). It follows from that equation that, in the large N limit, the joint probability distribution for $\psi(\mathbf{r}_1)\equiv\psi_1$ and $\psi(\mathbf{r}_2)\equiv\psi_2$ is

$$W(\psi_1,\psi_2)=\frac{1}{\pi^2\det K}\exp[-\psi_i^*(K^{-1})_{ij}\psi_j] \tag{14}$$

where $K_{ij}=\langle\psi_i\psi_j^*\rangle$ is the 2×2 covariance matrix with $K_{11}=K_{22}=1$, $K_{12}=K_{21}^*=f(\rho)$, and $\rho\equiv|\mathbf{r}_1-\mathbf{r}_2|$. The explicit form of the field-field correlation function $f(\rho)$ will be given below. Transforming to polar coordinates, $\psi_i=\sqrt{I_i}\exp(i\phi_i)$, and integrating out the phases, one obtains

$$P(I_1,I_2)=\frac{1}{1-|f|^2}\exp\left(-\frac{I_1+I_2}{1-|f|^2}\right)I_0\left(\frac{2|f|\sqrt{I_1I_2}}{1-|f|^2}\right). \tag{15}$$

Equation (14) is the standard assumption in the theory of optical and acoustical speckles and the resulting distribution $P(I_1,I_2)$ is well known in optics and acoustics of disordered media [1,2]. After averaging the product $\psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2)$ over the random phases θ_n, one finds $f(\rho)=\Sigma_n\exp(i\mathbf{k}_n\cdot\rho)$ where the sum is taken over N points on a unit sphere. In the $N\to\infty$ limit, replacing the sum by an integral, one finds $f(\rho)=J_0(k_0\rho)$ in two dimensions and $f(\rho)=(k_0\rho)^{-1}\sin(k_0\rho)$ in three dimensions. (A more rigorous calculation [3] shows that, for an open geometry, $f(\rho)$ decays exponentially for ρ larger than the mean free path ℓ.)

Equations (14) and (15) describe the statistics of radiation in an open system. In contrast, for a weakly coupled cavity (under resonance condition) the main part of the field corresponds to a standing wave. Such a field is represented by a sum of real waves, Eq. (8), and a derivation analogous to the one outlined above gives

$$P(I_1,I_2)=\frac{1}{2\pi\sqrt{1-f^2}}\frac{1}{\sqrt{I_1I_2}}\times\exp\left(-\frac{I_1+I_2}{2(1-f^2)}\right)\cosh\left(\frac{f\sqrt{I_1I_2}}{1-f^2}\right). \tag{16}$$

The statistics of eigenstates $\psi_\alpha(\mathbf{r})$ in a closed system has been rigorously studied by Prigodin and co-workers with the help of a zero-dimensional supersymmetric nonlinear σ model [10,11]. They studied the joint probability distribution $P(u_1,u_2)$, where $u_i\equiv|\psi_\alpha(\mathbf{r}_i)|^2$ $(i=1,2)$. For the unitary case [10] (broken time-reversal symmetry), an expression identical to Eq. (15) (with I_i replaced by u_i) was obtained. For the orthogonal case, Prigodin *et al.* [11] ended up with a rather complicated expression, containing a double integral. Later it was shown by Srednicki [12] that the expression in Ref. [11] can be reduced to the function given in Eq. (16). He was using the assumption [5] that a chaotic wave function, $\psi_\alpha(\mathbf{r})$, obeys the statistics of a Gaussian random process. This is in complete analogy with the standard assumption of the speckle theory [1,2], as outlined above. Again, the difference is that in the speckle theory one usually considers propagating waves in an open geometry, whereas Refs. [10–12] study a single eigenstate in an isolated system.

Now, we can analyze the general case of a disordered cavity coupled with arbitrary strength to the external world. The local field is now given by a combination of traveling and standing waves, Eq. (10). As a result, the real and imaginary parts of the field at two points, $\mathbf{r}_1$ and $\mathbf{r}_2$, are components of a four-dimensional Gaussian vector, $\Phi^T=(\mathrm{Re}\psi_1,\mathrm{Im}\psi_1,\mathrm{Re}\psi_2,\mathrm{Im}\psi_2)$, with the following covariance matrix:

$$K_{ij}\equiv\langle\Phi_i\Phi_j\rangle=\frac{1}{1+2\epsilon^2}\begin{bmatrix}1+\epsilon^2 & 0 & (1+\epsilon^2)f & 0\\ 0 & \epsilon^2 & 0 & \epsilon^2 f\\ (1+\epsilon^2)f & 0 & 1+\epsilon^2 & 0\\ 0 & \epsilon^2 f & 0 & \epsilon^2\end{bmatrix}. \tag{17}$$

After some lengthy algebra, this leads to

$$P_\delta(I_1,I_2)=\frac{\exp[-(I_1+I_2)/2\delta]}{2\delta(1-f^2)}\int_0^{2\pi}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\times\exp\left\{\frac{\sqrt{1-2\delta}}{2\delta(1-f^2)}\left[I_1\cos2\theta_1+I_2\cos2\theta_2+2f\sqrt{I_1I_2}\left(\cos(\theta_1-\theta_2)-\frac{\cos(\theta_1+\theta_2)}{\sqrt{1-2\delta}}\right)\right]\right\}. \tag{18}$$

Equation (18) interpolates between a weakly coupled cavity (at the resonance) and an open system.

In conclusion, we consider statistics of radiation in a disordered cavity coupled to the environment. The coupling can occur via an opening in the wall of the cavity or via absorption in the bulk. For weak coupling, and when the source frequency ω is close to an eigenfrequency ω_α, the wave generated in the cavity is close to a (standing) eigenmode $\psi_\alpha(\mathbf{r})$, with only a small admixture of a traveling wave. The intensity statistics is defined by the statistics of the eigenfunction $\psi_\alpha(\mathbf{r})$. For strong coupling the system becomes open and we recover the old results of the speckle theory for propagating waves. These results for the intensity distribution look very similar to the recently derived expressions for the eigenfunction amplitude distribution in closed systems with broken time-reversal symmetry. Thus, in optical systems considered in the present paper, coupling to the environment breaks the time-reversal symmetry either via the boundary conditions or by absorption in the bulk.

There are many similarities between intensity correlations in open random systems and correlations in a single eigenstate of a disordered cavity. There are also some differences. In open systems, the correlations described by Eq. (15) are valid for distances $\rho\lesssim\ell$, where ℓ is the mean free path. For distances $\rho\gg\ell$ a rather different type of correlation, due to diffusion, takes over [13].

Finally, let us mention that intensity distributions discussed above, such as in Eqs. (1), (9), or (13), apply only to the ''bulk'' of the distributions. Tails of the distributions, corresponding to very large or very small values of I, will show significant deviations from the above-given expressions and will not be universal. Indeed, it is well known that, both in open and closed systems, distributions for various quantities [conductance, density of states, $|\psi_\alpha(\mathbf{r})|^2$] develop log-normal tails [14]. This must also be true for the intensity distribution $P(I)$ discussed in this paper. For instance, for a point source placed at $\mathbf{r}=0$, the field $\psi_\omega(\mathbf{r})$ is just the Green's function $G_\omega(0,\mathbf{r})$ and the intensity is $I=|G_\omega(0,\mathbf{r})|^2$. The Green's function can be expanded in terms of the eigenfunctions $\psi_\alpha(\mathbf{r})$, and the log-normal tail of $P(|\psi_\alpha|^2)$ is responsible for such tails in the intensity distribution.

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